Flavour Symmetry Models after Daya Bay and RENO

Steve King
November 16th 2012,IFIC, Valencia

History of Neutrino Mixing (98-)

- ☑ Atmospheric ν_μ disappear, large θ_{23} (SK) (98)
- ☑ Solar ν_e disappear, large θ_{12} (H/S,Ga,SK) (02)
- ☑ Solar ν_e are converted to $\nu_\mu + \nu_\tau$ (SNO) (02)
- ☑ Reactor anti-ν_e disappear/reappear (KamLAND) (04)
- ☑ Accelerator ν_μ disappear (K2K 04 , MINOS 06)
- ☑ Accelerator ν_μ converted to ν_τ (OPERA 10)
- ☑ Accelerator ν_μ converted to ν_e , θ_{13} hint(T2K, MINOS,DC) (11)
- ☑ Reactor anti-ν_e disappear, θ_{13} meas. (Daya Bay, RENO) (12)

Implications for PP and Cosmology

- ☐ Origin of tiny neutrino mass

  Extra dimensions, See-saw mechanism, RPV SUSY

- ☐ Unification of matter, forces and flavour

  SUSY, GUTs, Family Symmetry,...

Particle Physics

- ☐ Did neutrinos play a role in our existence?

  Leptogenesis

- ☐ Did neutrinos play a role in forming galaxies?

  Hot/Warm Dark matter component

- ☐ Did neutrinos play a role in birth of the universe?

  Sneutrino inflation

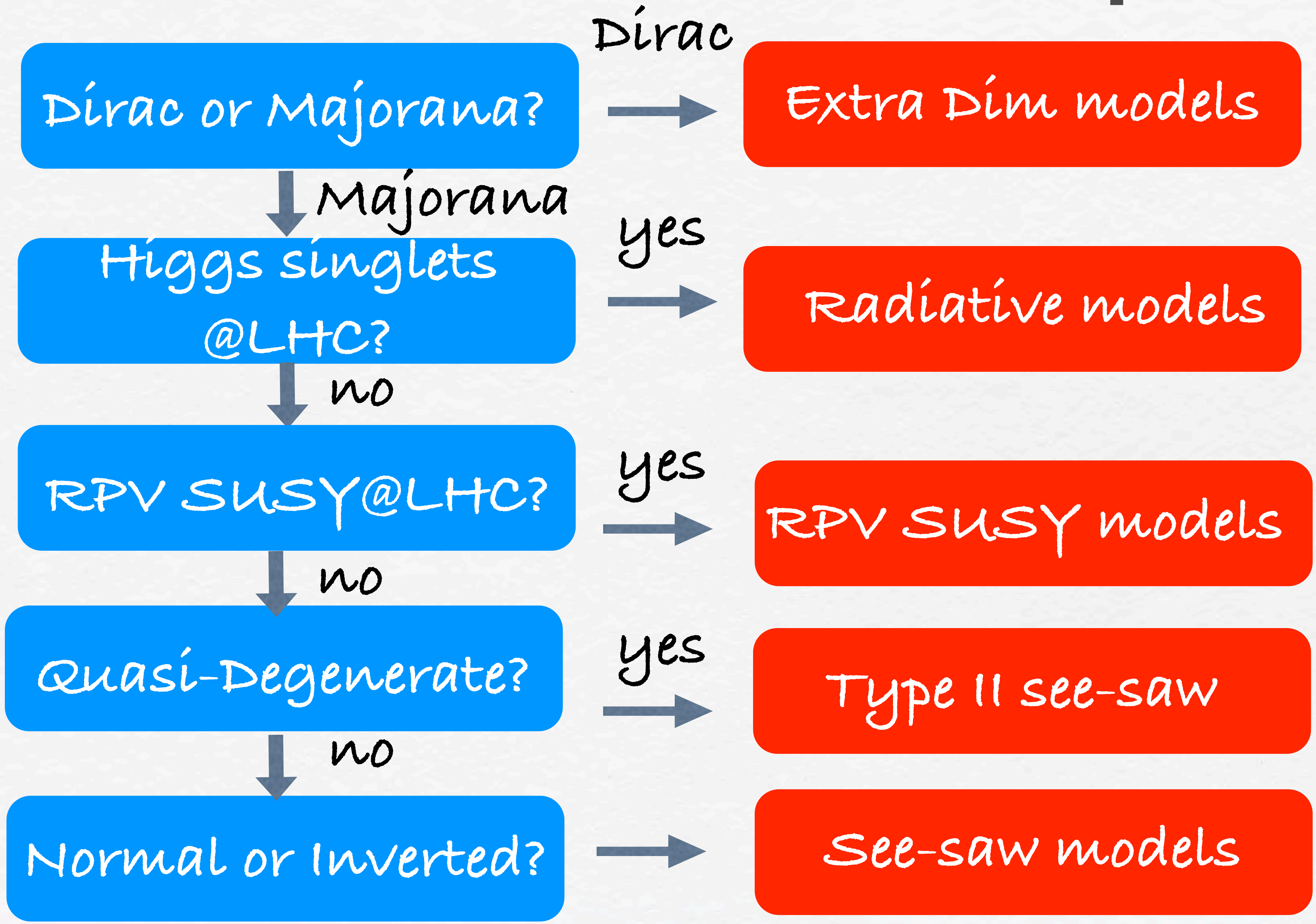
- ☐ Can neutrinos shed light on dark energy? $\Lambda \sim m_\nu^4$

Cosmology

Origin of Neutrino Mass

- Neutrino mass is the first (and so far only) new physics beyond the Standard Model
- Lepton Flavor is not conserved: L_e, L_μ, L_τ broken
- Neutrino mass may be Dirac or Majorana
- The Origin of neutrino mass is unknown
- Roadmap of possible mass mechanisms

Neutrino mass roadmap



See-saw mechanism

Possible type II contribution

Dirac matrix

$$\left(\overline{\nu_L} \quad \overline{\nu_R^c}\right)\begin{pmatrix} 0 & m_{LR} \\ m_{LR}^T & M_{RR} \end{pmatrix}\begin{pmatrix} \nu_L^c \\ \nu_R \end{pmatrix}$$

$$M^{\nu} = m_{LR} \cdot \frac{1}{M_{RR}} \cdot m_{LR}^T$$

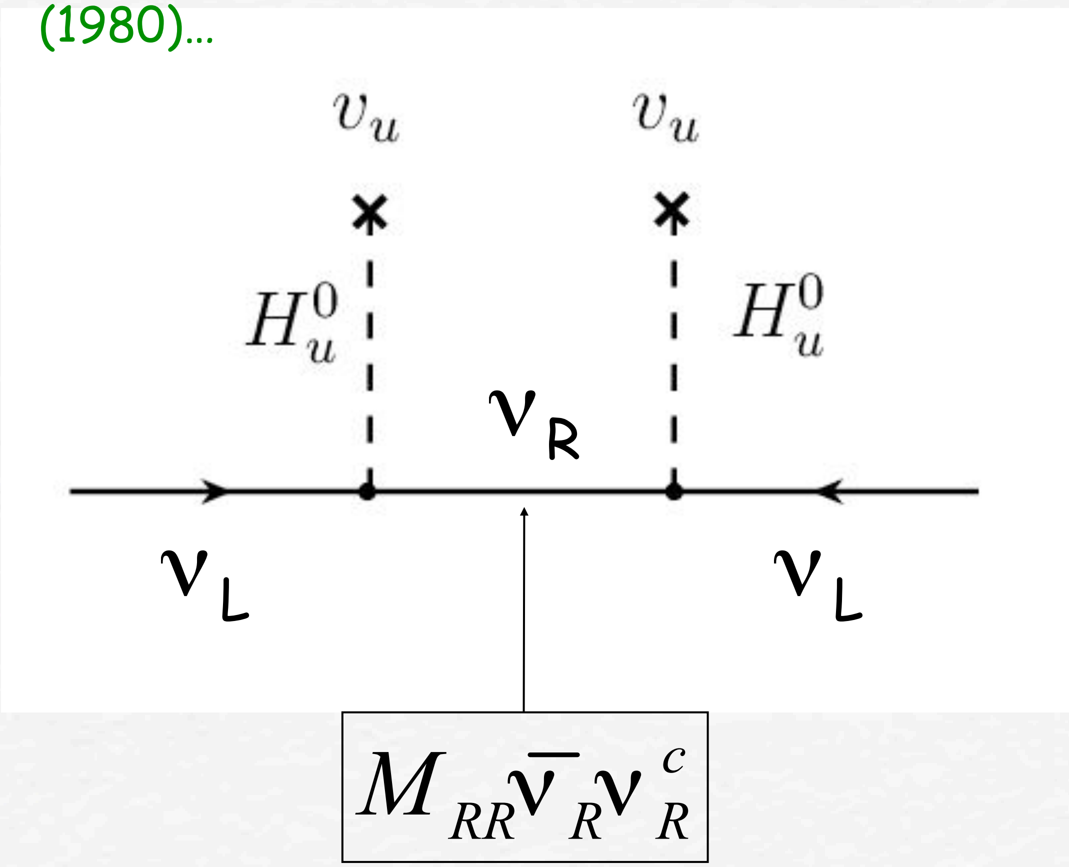
Light Majorana matrix

Heavy Majorana matrix

- ☐ Neutrinos are light because RH neutrinos are heavy
- ☐ Allows large neutrino mixing

Type I see-saw mechanism

P. Minkowski (1977), Gell-Mann, Glashow, Mohapatra, Ramond, Senjanovic, Slanski, Yanagida (1979/1980), Schechter and Valle (1980)...

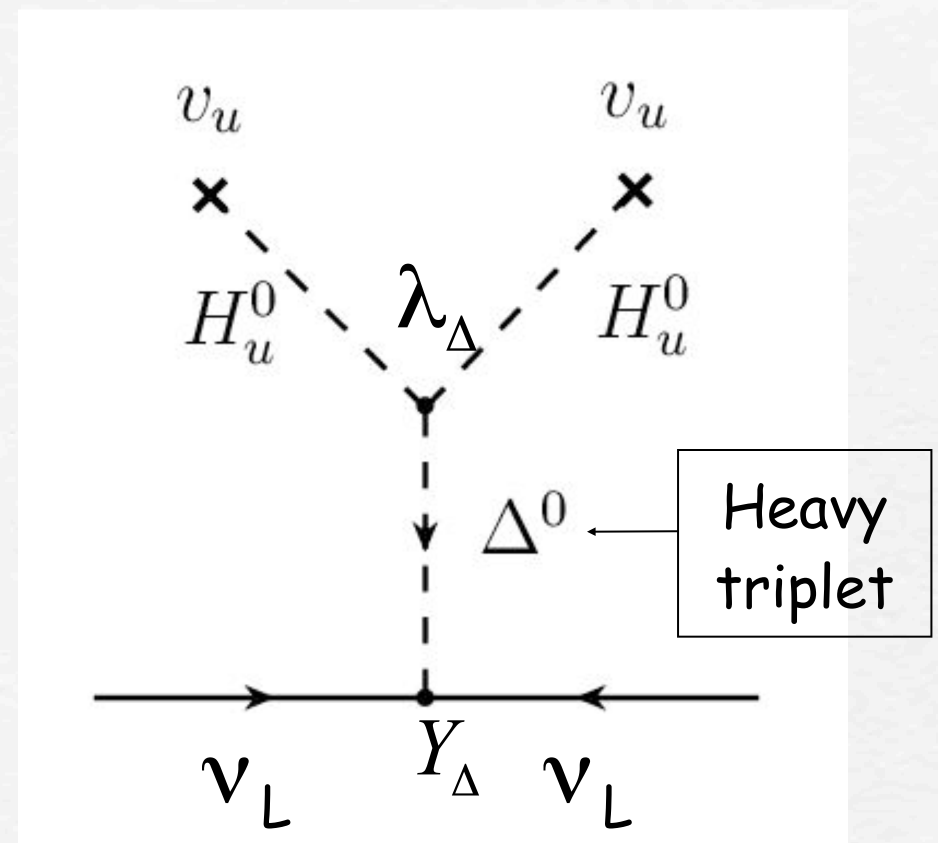


$$m_{LL}^{I} \approx -m_{LR} M_{RR}^{-1} m_{LR}^{T}$$

Type I

Type II see-saw mechanism (SUSY)

Lazarides, Magg, Mohapatra, Senjanovic, Shafi, Wetterich, Schechter and Valle...



$$m_{LL}^{II} \approx \lambda_\Delta Y_\Delta \frac{v_u^2}{M_\Delta}$$

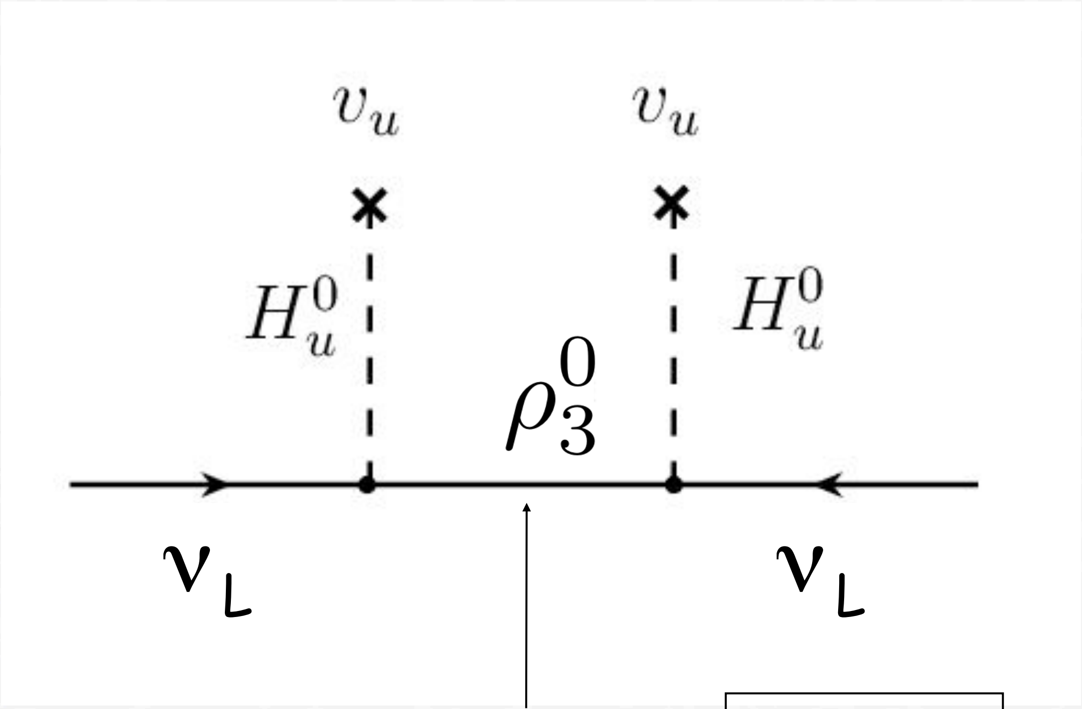
Type II

Type III see-saw mechanism

Foot, Lew, He, Joshi; Ma…

Supersymmetric adjoint SU(5)

Perez et al; Cooper, SFK, Luhn,…



SU(2)$_L$ fermion triplet

$$\boxed{M_\rho \rho\rho}$$

$$\boxed{m_{LL}^{III} \approx -m_{LR} M_\rho^{-1} m_{LR}^T}$$

Type III

See-saw mechanisms with extra singlets S

Inverse see-saw

Wyler, Wolferstein; Mohapatra, Valle

$$\begin{pmatrix} 0 & M_D & 0 \\ M_D^T & 0 & M \\ 0 & M^T & \mu \end{pmatrix} \qquad M_\nu = M_D M^{T^{-1}} \mu M^{-1} M_D^T$$

M ≈ TeV ➔ LHC

Linear see-saw

Malinsky, Romao, Valle

$$\begin{pmatrix} 0 & M_D & M_L \\ M_D^T & 0 & M \\ M_L^T & M^T & 0 \end{pmatrix}$$

$$M_\nu = M_D (M_L M^{-1})^T + (M_L M^{-1}) M_D{}^T$$

LFV predictions

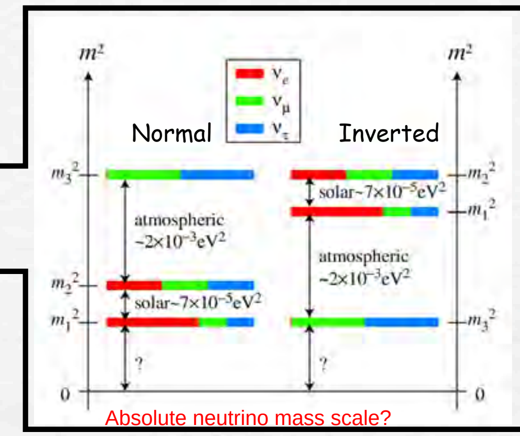
Three Neutrino Mixing

Standard Model states

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L \quad \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L \quad \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L$$

PMNS matrix

Neutrino mass states



Pontecorvo
Maki
Nakagawa
Sakata

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$

$$U_{PMNS} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{i\alpha_1/2} & 0 & 0 \\ 0 & e^{i\alpha_2/2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$s_{ij} = \sin\theta_{ij}$

$c_{ij} = \cos\theta_{ij}$

Atmospheric | Reactor | Solar | Majorana

Oscillation phase δ

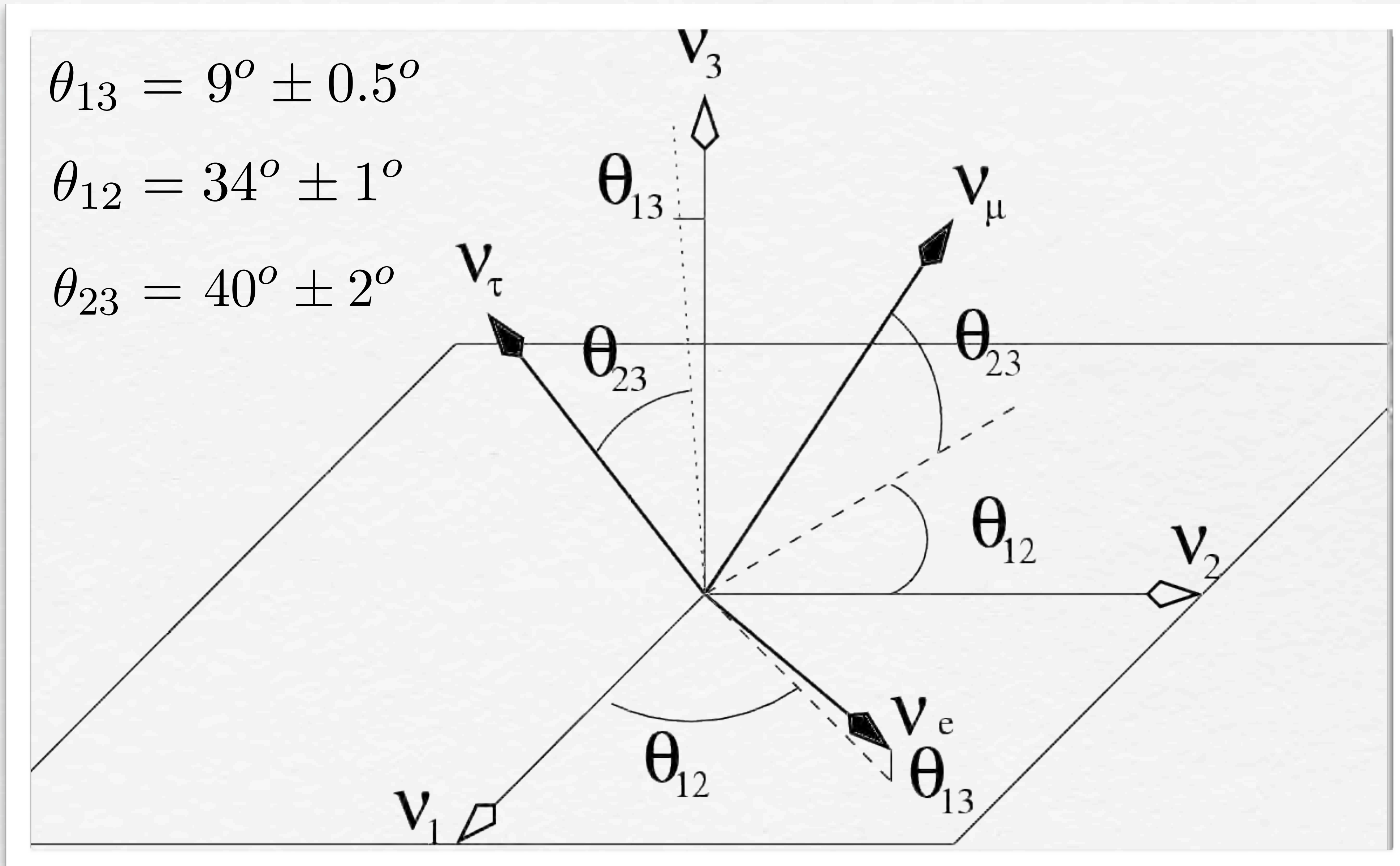
Majorana phases α_1, α_2

3 masses + 3 angles + 1 (or 3) phase(s)
= 7 (or 9) new parameters for SM

Global Fits 2012

parameter	Tortola et al	Fogli et al	Schwetz et al
$\Delta m^2_{21}[10^{-5}\mathrm{eV}^2]$	7.62 ± 0.19	$7.54^{+0.26}_{-0.22}$	$7.50^{+0.185}_{-0.185}$
$\Delta m^2_{31}[10^{-3}\mathrm{eV}^2]$	$2.55^{+0.06}_{-0.09}$ $-2.43^{+0.06}_{-0.07}$	$2.43^{+0.07}_{-0.09}$ $-2.42^{+0.09}_{-0.07}$	$2.47^{+0.069}_{-0.067}$ $-2.43^{+0.042}_{-0.065}$
$\sin^2\theta_{12}$	$0.320^{+0.016}_{-0.017}$	$0.307^{+0.018}_{-0.016}$	$0.30^{+0.013}_{-0.013}$
$\sin^2\theta_{23}$	$0.427^{+0.034}_{-0.027}$ $0.600^{+0.0026}_{-0.031}$	$0.386^{+0.024}_{-0.021}$ $0.392^{+0.039}_{-0.022}$	$0.41^{+0.037}_{-0.025}$ $(0.59^{+0.021}_{-0.022})$
$\sin^2\theta_{13}$	$0.0246^{+0.0029}_{-0.0028}$ $0.0250^{+0.0026}_{-0.0031}$	$0.0241^{+0.0025}_{-0.0025}$ $0.0244^{+0.0023}_{-0.0025}$	0.023 ± 0.0023
δ	$(0.80 \pm 1)\pi$ $-(0.03 \pm 1)\pi$	$(1.08^{+0.24}_{-0.31})\pi$ $(1.09^{+0.38}_{-0.26})\pi$	$(1.67^{+0.37}_{-0.77})\pi$

Neutrino Mixing Angles



Simple LO mixing patterns $\theta_{13} = 0 \;\; \theta_{23} = 45^o$

- **Bimaximal**

V. Barger, S. Pakvasa, T. Weiler and K. Whisnant

$$U_{BM} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix} P \quad \theta_{12} = 45^o$$

- **Tri-bimaximal**

Harrison, Perkins and Scott

$$U_{TB} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} P \quad \theta_{12} = 35.26^o$$

- **Golden ratio**

Datta, Ling, Ramond; Kajirama, Raidal, Strumia; Everett, Stuart,Ding: Feruglio, Paris

$$U_{GR} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -\frac{s_{12}}{\sqrt{2}} & -\frac{c_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{s_{12}}{\sqrt{2}} & -\frac{c_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} P$$

$$\phi = \frac{1+\sqrt{5}}{2} \qquad \tan\theta_{12} = \frac{1}{\phi} \qquad \theta_{12} = 31.7^o$$

King 1205.0506 Physics Letters B 718 (2012)

Data prefers Tri-bimaximal-Cabibbo Mixing

Combine TB mixing with $\theta_{13} \approx \frac{\theta_C}{\sqrt{2}} \approx 9.2^o$

$$s_{13} = \frac{\lambda}{\sqrt{2}}, \quad s_{12} = \frac{1}{\sqrt{3}}, \quad s_{23} = \frac{1}{\sqrt{2}} \qquad \lambda = 0.2253 \pm 0.0007$$

$$U_{TBC} \approx \begin{pmatrix} \sqrt{\frac{2}{3}}(1-\frac{1}{4}\lambda^2) & \frac{1}{\sqrt{3}}(1-\frac{1}{4}\lambda^2) & \frac{1}{\sqrt{2}}\lambda e^{-i\delta} \\ -\frac{1}{\sqrt{6}}(1+\lambda e^{i\delta}) & \frac{1}{\sqrt{3}}(1-\frac{1}{2}\lambda e^{i\delta}) & \frac{1}{\sqrt{2}}(1-\frac{1}{4}\lambda^2) \\ \frac{1}{\sqrt{6}}(1-\lambda e^{i\delta}) & -\frac{1}{\sqrt{3}}(1+\frac{1}{2}\lambda e^{i\delta}) & \frac{1}{\sqrt{2}}(1-\frac{1}{4}\lambda^2) \end{pmatrix} + \mathcal{O}(\lambda^3)$$

Approximate description of lepton mixing

Hints of a connection with quark mixing

King; Parke; Pakvasa, Rodejohann, Weiler

useful to expand PMNS about TB mixing

$$U_{\rm PMNS} \approx \begin{pmatrix} \frac{2}{\sqrt{6}}(1-\frac{1}{2}s) & \frac{1}{\sqrt{3}}(1+s) & \frac{1}{\sqrt{2}}re^{-i\delta} \\ -\frac{1}{\sqrt{6}}(1+s-a+re^{i\delta}) & \frac{1}{\sqrt{3}}(1-\frac{1}{2}s-a-\frac{1}{2}re^{i\delta}) & \frac{1}{\sqrt{2}}(1+a) \\ \frac{1}{\sqrt{6}}(1+s+a-re^{i\delta}) & -\frac{1}{\sqrt{3}}(1-\frac{1}{2}s+a+\frac{1}{2}re^{i\delta}) & \frac{1}{\sqrt{2}}(1-a) \end{pmatrix} P$$

$$\sin\theta_{12} = \frac{1}{\sqrt{3}}(1+s)\ , \qquad \sin\theta_{23} = \frac{1}{\sqrt{2}}(1+a)\ , \qquad \sin\theta_{13} = \frac{r}{\sqrt{2}}$$

$$s = -0.03 \pm 0.03, \qquad a = -0.10 \pm 0.03, \qquad r = 0.22 \pm 0.01$$

s = solar
below Tri-max

a = atmospheric
below Bi-max

r = reactor
Cabibbo-like

Global fits hint at atmospheric angle in first octant
i.e. $a<0$

Tri-bimaximal Variants

- General $(s, a, r \neq 0)$

$$U_{\mathrm{PMNS}} \approx \begin{pmatrix} \frac{2}{\sqrt{6}}(1-\frac{1}{2}s) & \frac{1}{\sqrt{3}}(1+s) & \frac{1}{\sqrt{2}}re^{-i\delta} \\ -\frac{1}{\sqrt{6}}(1+s-a+re^{i\delta}) & \frac{1}{\sqrt{3}}(1-\frac{1}{2}s-a-\frac{1}{2}re^{i\delta}) & \frac{1}{\sqrt{2}}(1+a) \\ \frac{1}{\sqrt{6}}(1+s+a-re^{i\delta}) & -\frac{1}{\sqrt{3}}(1-\frac{1}{2}s+a+\frac{1}{2}re^{i\delta}) & \frac{1}{\sqrt{2}}(1-a) \end{pmatrix} P$$

- Tri-bimaximal-reactor $(s=a=0)$

$$U_{TBR} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}re^{-i\delta} \\ -\frac{1}{\sqrt{6}}(1+re^{i\delta}) & \frac{1}{\sqrt{3}}(1-\frac{1}{2}re^{i\delta}) & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}}(1-re^{i\delta}) & -\frac{1}{\sqrt{3}}(1+\frac{1}{2}re^{i\delta}) & \frac{1}{\sqrt{2}} \end{pmatrix} P$$

King; Antusch,Boudjemaa,King; Morisi,Patel,Peinado; Luhn, King

- Tri-maximal 1 $(s=0,\ a=r.\cos\delta)$

$$U_{\mathrm{TM}_1} = P' \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}re^{-i\delta} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}(1-\frac{3}{2}re^{i\delta}) & \frac{1}{\sqrt{2}}(1+re^{-i\delta}) \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}(1+\frac{3}{2}re^{i\delta}) & -\frac{1}{\sqrt{2}}(1-re^{-i\delta}) \end{pmatrix} P$$

Lam; Albright,Rodejohann; Antusch, King, Luhn, Spinrath

Haba,Watanabe,Yoshioka; He,Zee; Grimus,Lavoura; Albright,Rodejohann; King, Luhn

- Tri-maximal 2 $(s=0,\ a=-r/2.\cos\delta)$

$$U_{\mathrm{TM}_2} = P' \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}re^{-i\delta} \\ -\frac{1}{\sqrt{6}}(1+\frac{3}{2}re^{i\delta}) & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}(1-\frac{1}{2}re^{-i\delta}) \\ -\frac{1}{\sqrt{6}}(1-\frac{3}{2}re^{i\delta}) & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}}(1+\frac{1}{2}re^{-i\delta}) \end{pmatrix} P$$

N.B. Atmospheric sum rules: $a=r.\cos\delta$, $a=-r/2.\cos\delta$

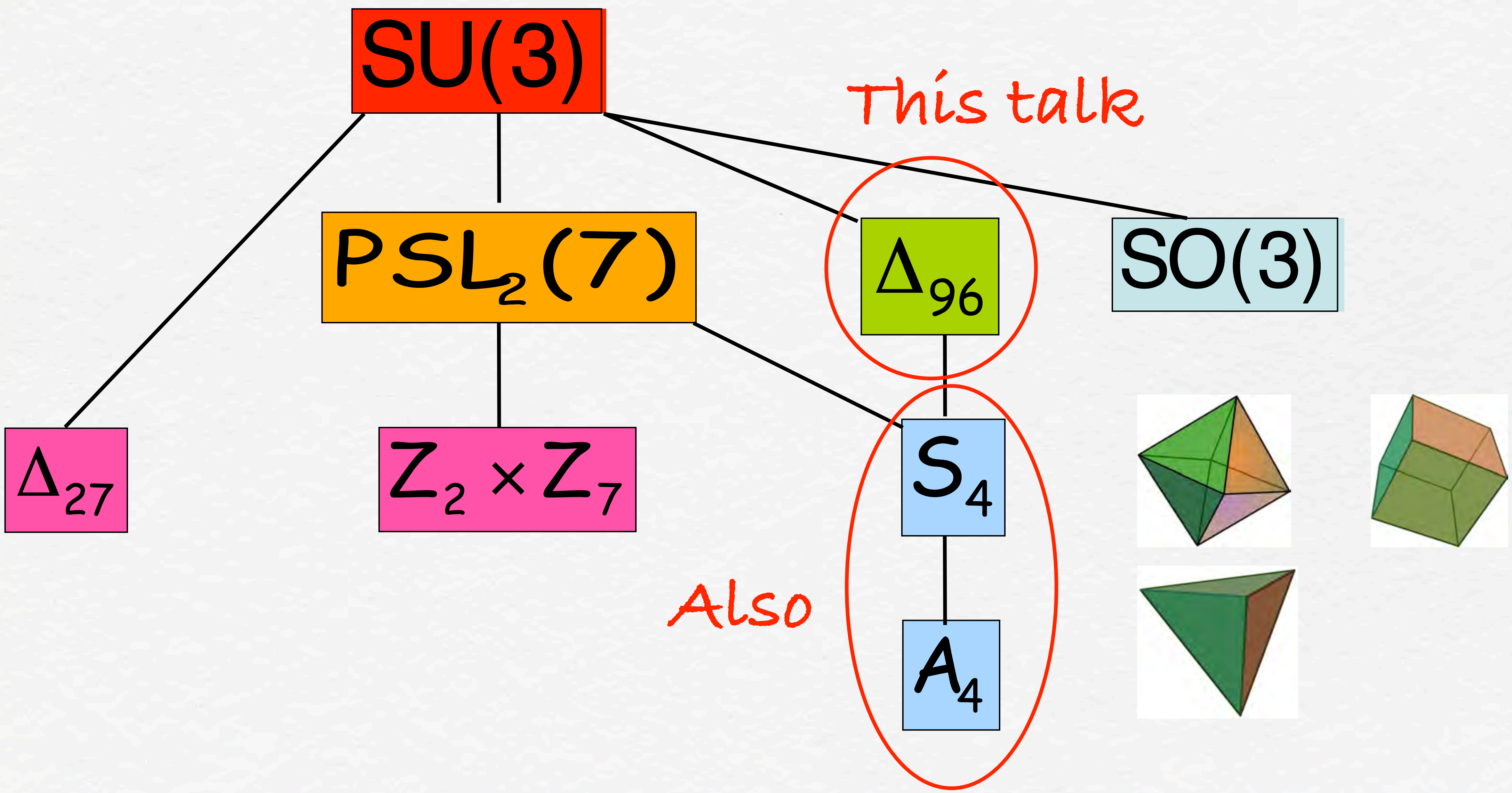
Theory Road Map

Daya Bay/RENO

- **Family Symmetry**
  - Indirect
  - Direct
- **Anarchy**
  - Landscape

Family Symmetry G

Family Symmetries G
which contain triplet reps
(three families in a triplet)



Indirect Approach

Family Symmetry

$$G$$

Charged Leptons Diagonal

G fully broken by flavon vevs

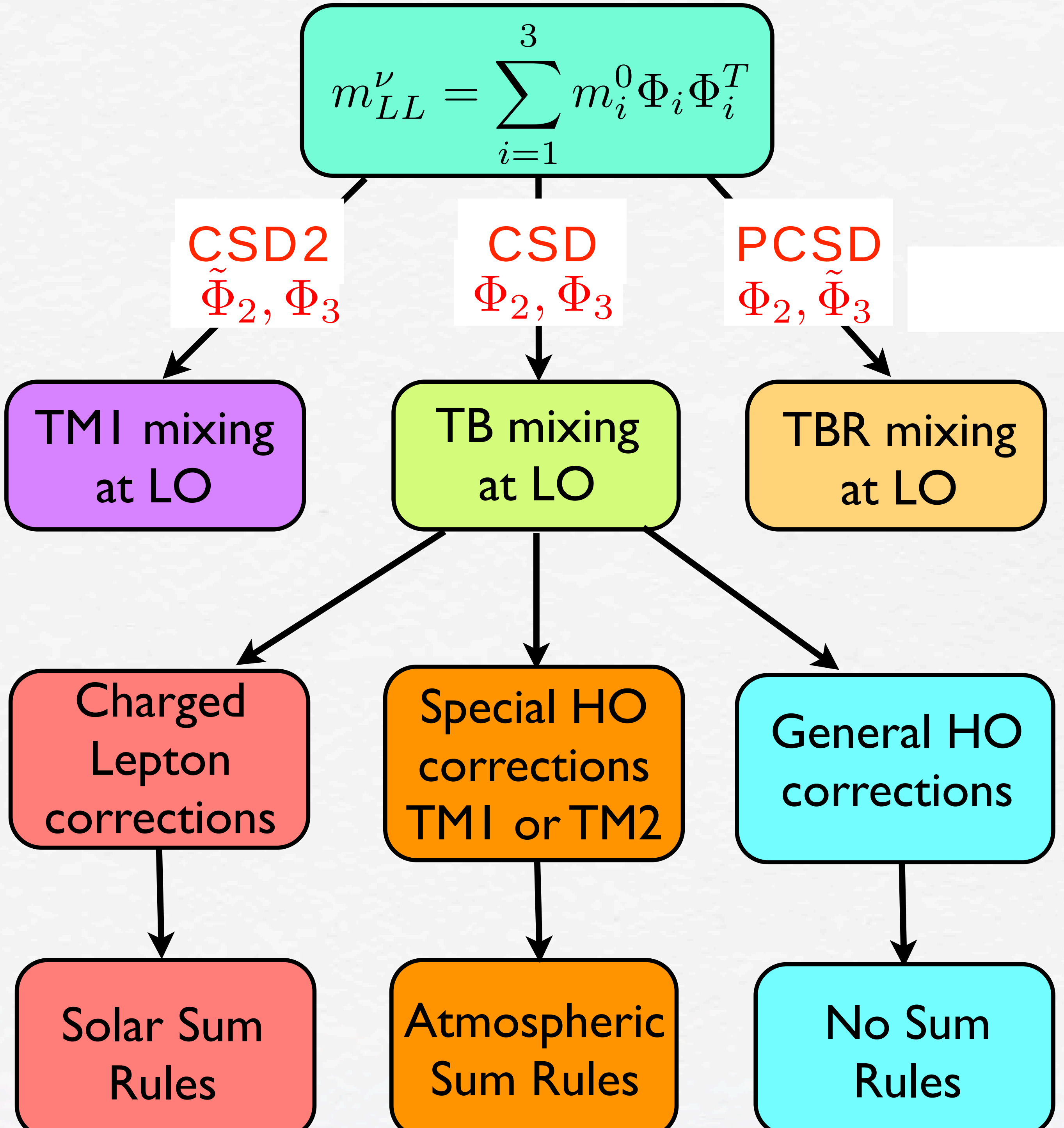
$$\mathcal{L}^{\nu} \sim L^T \sum_{i=1}^{3} \left(\frac{\langle \phi_i^{\nu} \rangle}{\Lambda} \cdot \frac{1}{M_i} \cdot \frac{\langle {\phi_i^{\nu}}^T \rangle}{\Lambda} \right) L H_u H_u$$

Special alignments

$$\frac{\langle \phi_i^{\nu} \rangle}{\Lambda} \propto \Phi_i$$

$$m_{LL}^{\nu} = \sum_{i=1}^{3} m_i^0 \Phi_i \Phi_i^T$$

Indirect Models after Daya Bay and RENO



Indirect Models after Daya Bay and RENO

King; Antusch,Boudjemaa; King,Luhn

$$m_{LR} = \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix}$$

PCSD

$$m^\nu = \underbrace{\cancel{\frac{AA^T}{M_1}}}_{0} + \frac{BB^T}{M_2} + \frac{CC^T}{M_3}$$

dominant

$$B \propto \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad C \propto \begin{pmatrix} re^{-i\delta} \\ 1 \\ -1 \end{pmatrix}$$

$$U_{TBR} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} re^{-i\delta} \\ -\frac{1}{\sqrt{6}}(1+re^{i\delta}) & \frac{1}{\sqrt{3}}(1-\frac{1}{2}re^{i\delta}) & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}}(1-re^{i\delta}) & -\frac{1}{\sqrt{3}}(1+\frac{1}{2}re^{i\delta}) & \frac{1}{\sqrt{2}} \end{pmatrix} P$$

$$s = a = 0, \;\; r \neq 0$$

Special case TBC: $r = \lambda$

Antusch,King,Luhn,Spinrath

$$m_{LR} = \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix}$$

CSD2

$$m^\nu = \underbrace{\cancel{\frac{AA^T}{M_1}}}_{0} + \frac{BB^T}{M_2} + \frac{CC^T}{M_3}$$

dominant

$$B \propto \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \qquad C \propto \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Two texture zeroes - Occam's razor?

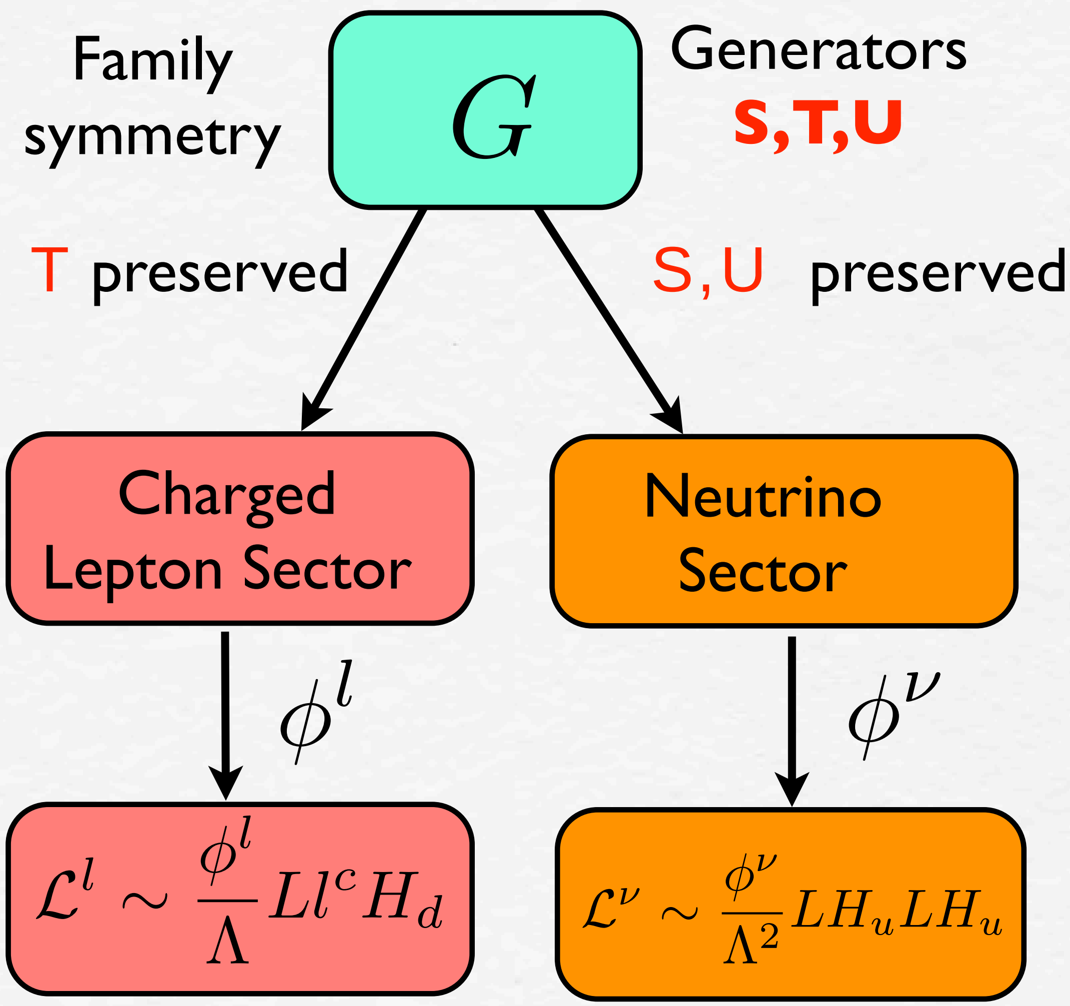
Harigaya, Ibe, Yanagida

$$U_{\mathrm{TM}_1} = P' \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} re^{-i\delta} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}(1-\frac{3}{2}re^{i\delta}) & \frac{1}{\sqrt{2}}(1+re^{-i\delta}) \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}(1+\frac{3}{2}re^{i\delta}) & -\frac{1}{\sqrt{2}}(1-re^{-i\delta}) \end{pmatrix} P$$

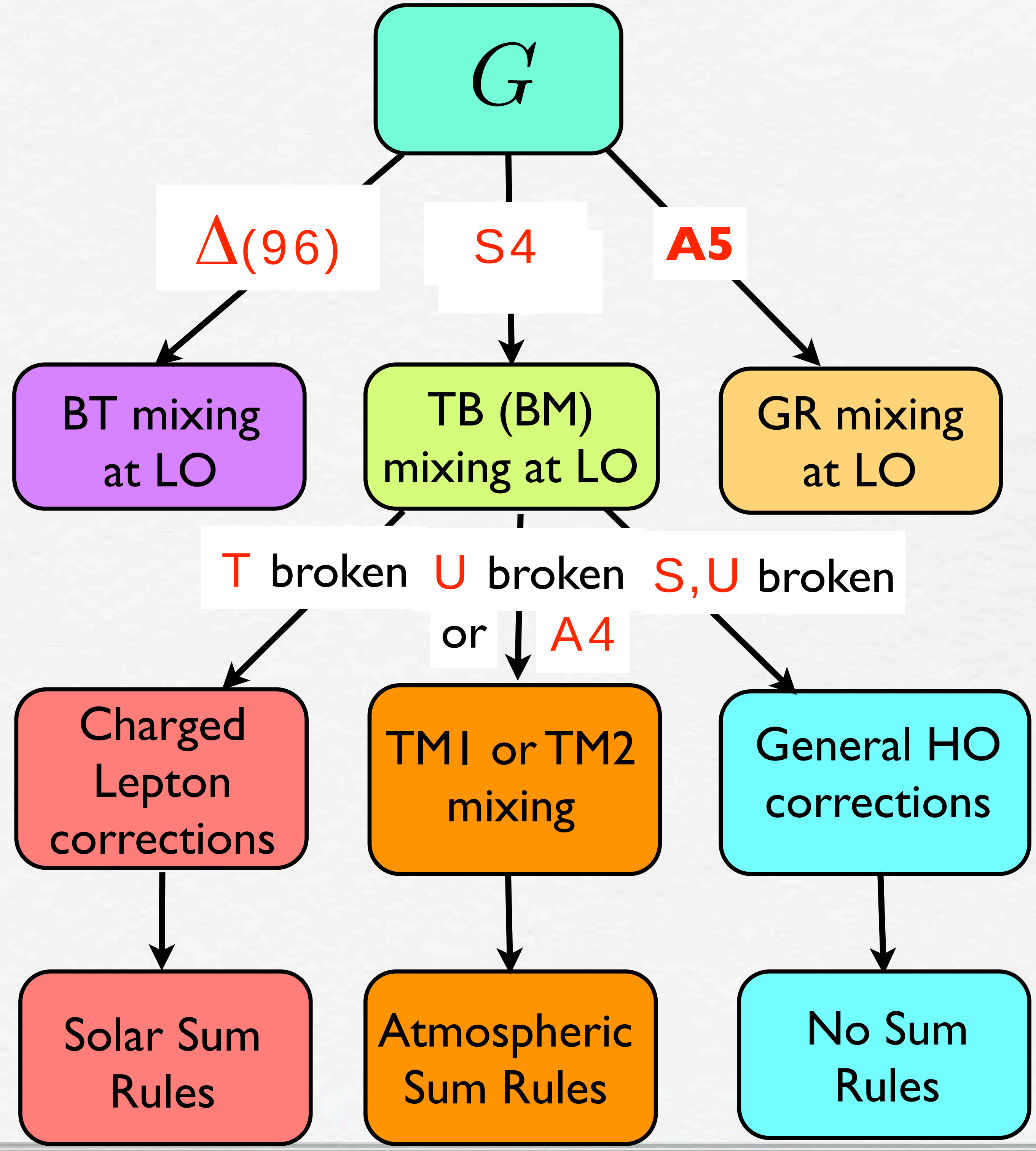
$$a = r\cos\delta \qquad s = 0$$

Atmospheric sum rule

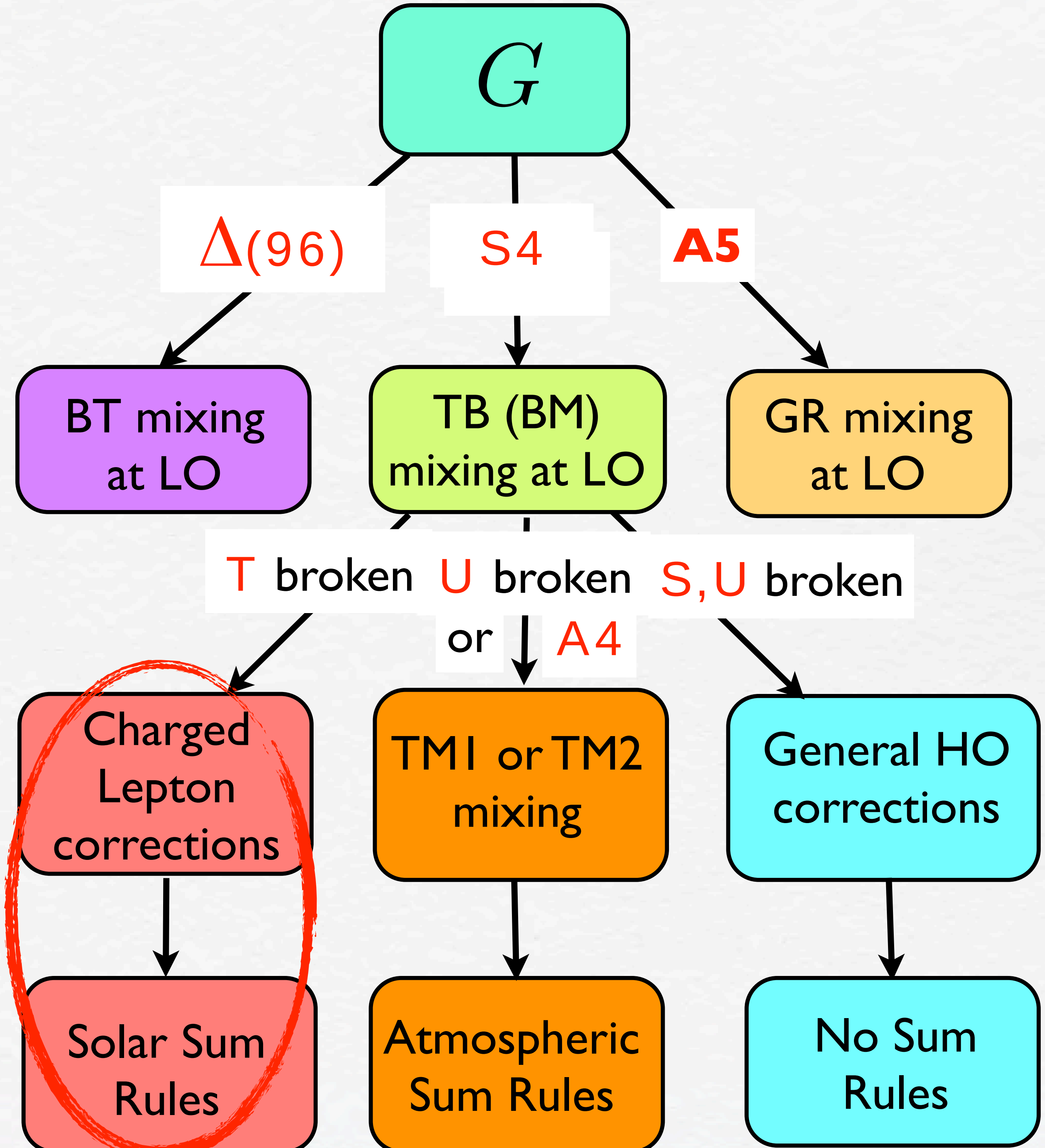
Direct Approach



Direct Models after Daya Bay and RENO



Direct Models after Daya Bay and RENO



Solar Sum Rule s=r.cosδ

King ('05); Masina ('05); Antusch, King ('05)
Antusch, Maurer ('11) Mazocca, Petcov, Romanino, Spinrath ('11) King 1205.0506
Antusch,Gross, Maurer, Sluka 1205.1051;

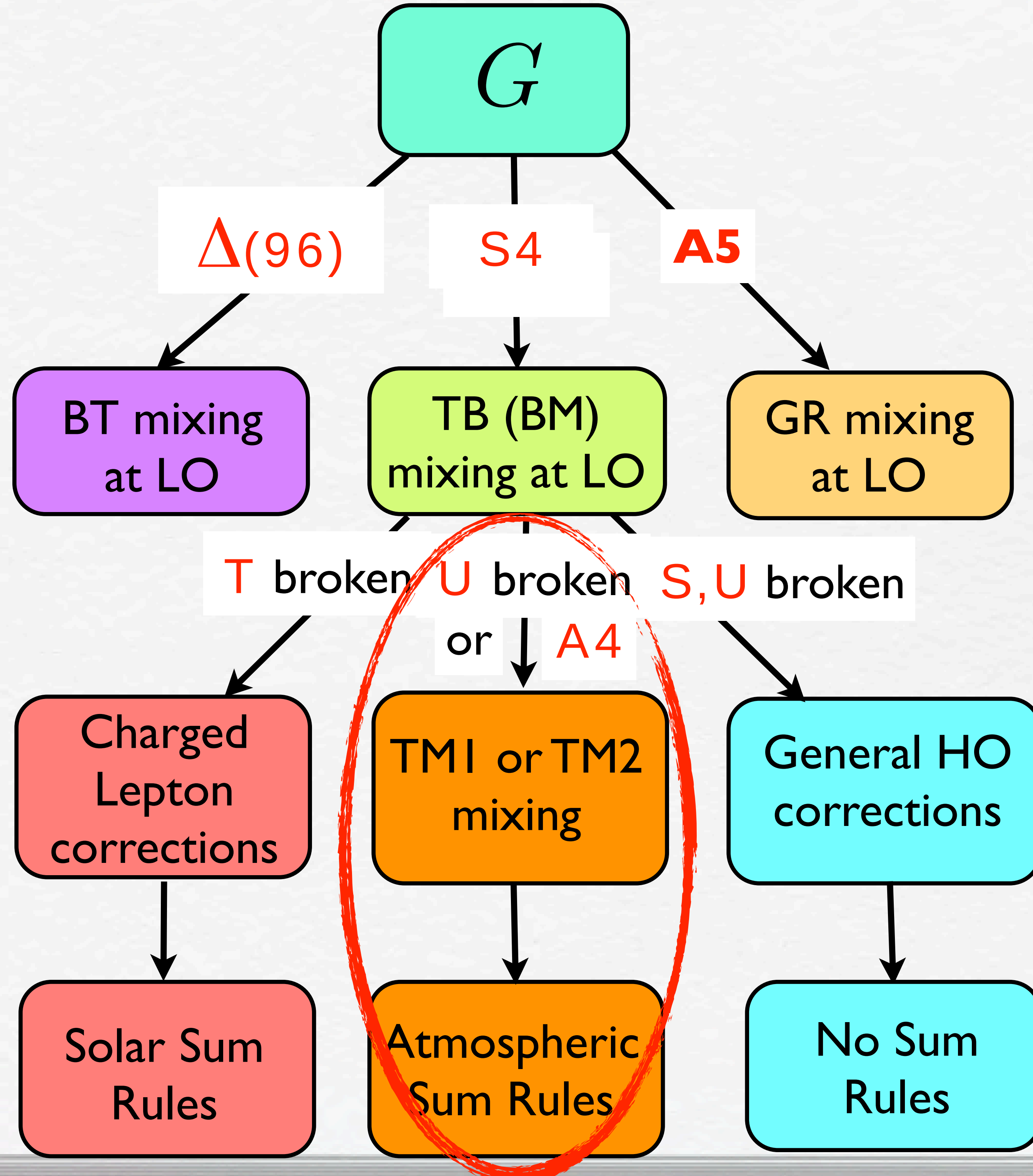
$$U_{PMNS} = V^e V^{\nu\dagger} = \underbrace{\begin{pmatrix} 1 & \lambda e^{-i\delta} & 0 \\ -\lambda e^{i\delta} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{Cabibbo-like}} \underbrace{\begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}}_{\text{Tri-bimaximal}}$$

$$\lambda = \text{Wolfenstein}$$

$$\sin\theta_{13} = \frac{\lambda}{\sqrt{2}}$$

$$U_{PMNS} \approx \begin{pmatrix} \sqrt{\frac{2}{3}}(1-(\lambda/2)\cos\delta) & \frac{1}{\sqrt{3}}(1+\overbrace{\lambda\cos\delta}^{s}) & \frac{1}{\sqrt{2}}\overbrace{\lambda}^{r} e^{-i\delta} \\ -\frac{1}{\sqrt{6}}(1+(2\lambda)\cos\delta) & \frac{1}{\sqrt{3}}(1-\lambda\cos\delta) & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Direct Models after Daya Bay and RENO



A4 Model

L=3, N^c=3, Hu=1

Cooper, King, Luhn;

Altarelli, Ferguglio,... Shimizu,Tanimoto,Watanabe

$$W^\nu_{A_4} = yLH_uN^c + (y_1\underbrace{\varphi_S}_{3} + y_2\underbrace{\xi}_{1} + y_3'\underbrace{\xi'}_{1'} + y_3''\underbrace{\xi''}_{1''})N^cN^c$$

$$m_D = \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix} y\,v_u \qquad \langle\varphi_S\rangle = v_S\begin{pmatrix}1\\1\\1\end{pmatrix}$$

TB violation $\Delta = \frac{1}{2}(\gamma'' - \gamma')$

$$M_R = \left[\alpha\begin{pmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{pmatrix} + \beta\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix} + \gamma'\begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix} + \gamma''\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}\right]$$

$$\alpha = y_1 v_S,\ \beta = y_2\langle\xi\rangle,\ \gamma' = y_3'\langle\xi'\rangle,\ \gamma'' = y_3''\langle\xi''\rangle$$

without ξ', ξ'' m_D, M_R respect S and U invariance → TB mixing

with ξ', ξ'' M_R respects S but not U invariance → TM2 mixing

□ S4 Model

Hagedorn, King, Luhn

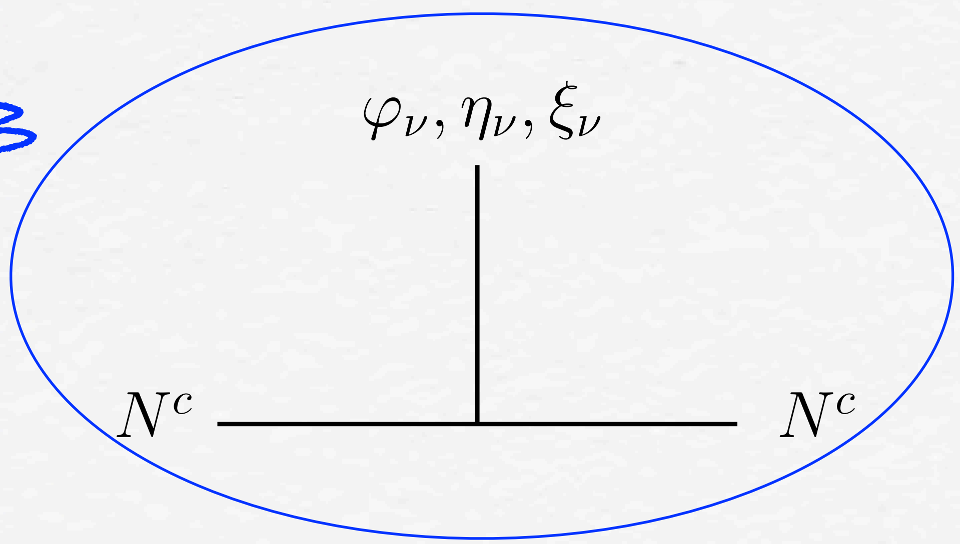
$$W_{S_4}^{\nu,\mathrm{eff}} \sim LH_uN^c + (\overset{3'}{\varphi_\nu} + \overset{1}{\xi_\nu} + \overset{2}{\eta_\nu})N^cN^c + \frac{\overset{1'}{\zeta_\nu}}{M_\chi}\overset{2}{\eta_\nu}N^cN^c$$

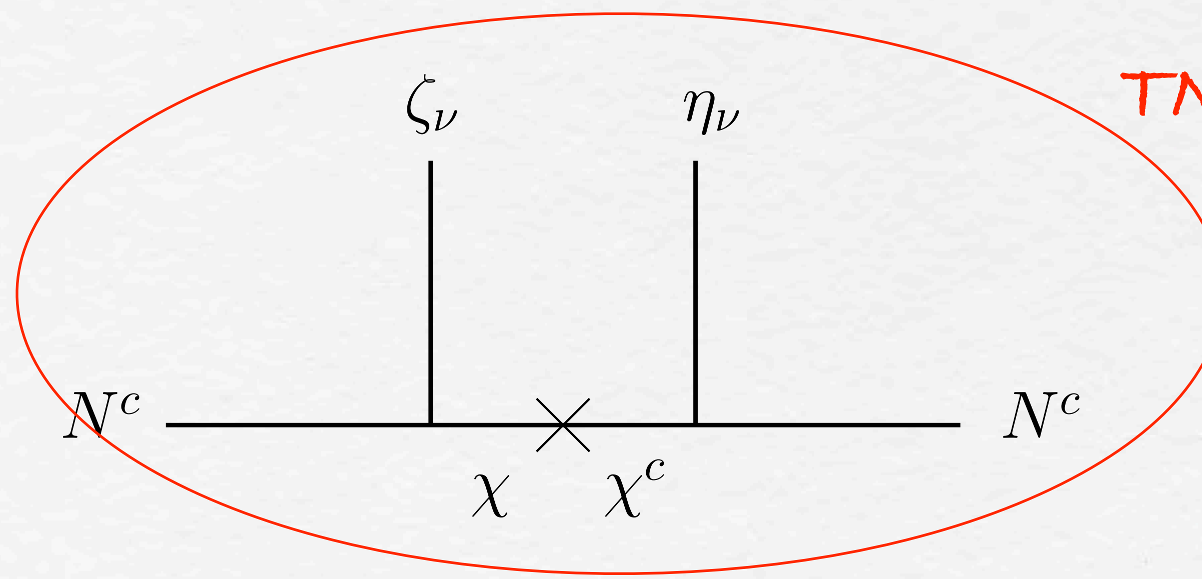
unifies ξ', ξ'' into a doublet η, restores TB

1'

$< \zeta_\nu >$ breaks S4→A4 and TB→TM2

TB



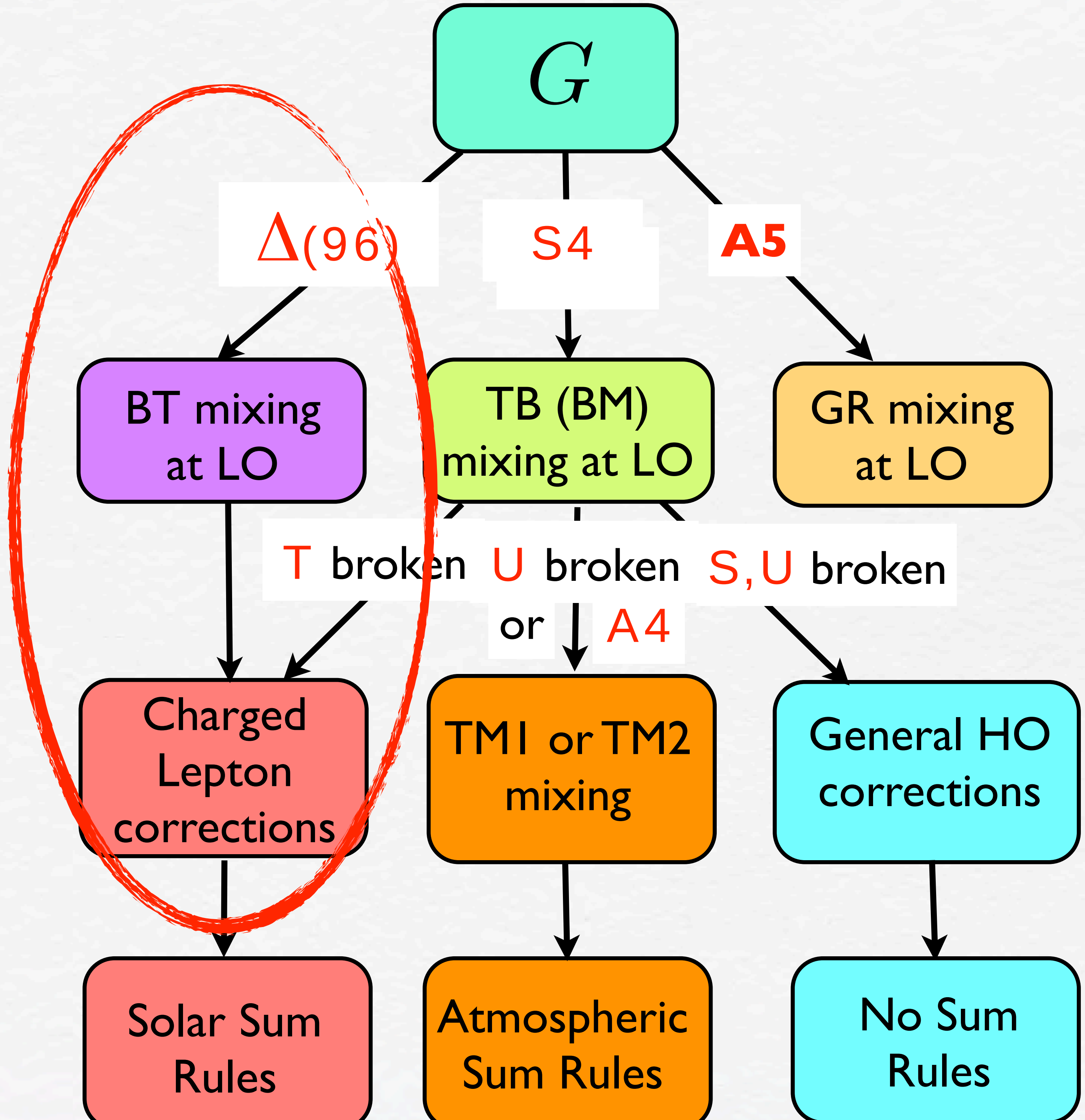


TM2

S4 model has TB at LO with TM2 corrections at HO

Explains why $\theta_{13} < \theta_{12}, \theta_{23}$

Direct Models after Daya Bay and RENO



$\Delta(96)$

$\Delta(6n^2)$ group with $n = 4$

$\Delta(96) \cong (Z_4 \times Z_4) \rtimes S_3$

$$\begin{gathered}
\mathbf{1} \otimes \mathbf{x} = \mathbf{x} \text{ with } \mathbf{x} \text{ any } \Delta(96) \text{ irrep} \\
\mathbf{1}' \otimes \mathbf{1}' = \mathbf{1} \\
\mathbf{1}' \otimes \mathbf{2} = \mathbf{2} \\
\mathbf{1}' \otimes \mathbf{r} = \mathbf{r}' \text{ when } \mathbf{r} = \mathbf{3},\ \widetilde{\mathbf{3}},\text{ or } \overline{\mathbf{3}} \\
\mathbf{1}' \otimes \mathbf{r}' = \mathbf{r} \text{ when } \mathbf{r} = \mathbf{3},\ \widetilde{\mathbf{3}},\text{ or } \overline{\mathbf{3}} \\
\mathbf{1}' \otimes \mathbf{6} = \mathbf{6} \\
\mathbf{2} \otimes \mathbf{2} = \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{2} \\
\mathbf{2} \otimes \mathbf{r}^m = \mathbf{r} \oplus \mathbf{r}' \text{ when } \mathbf{r} = \mathbf{3},\ \widetilde{\mathbf{3}},\text{ or } \overline{\mathbf{3}} \\
\mathbf{2} \otimes \mathbf{6} = \mathbf{6} \oplus \mathbf{6} \\
\mathbf{3}^m \otimes \mathbf{3}^n = \widetilde{\mathbf{3}}^p \oplus \overline{\mathbf{3}}' \oplus \overline{\mathbf{3}} \\
\mathbf{3}^m \otimes \widetilde{\mathbf{3}}^n = \overline{\mathbf{3}}^p \oplus \mathbf{6} \\
\mathbf{3}^m \otimes \overline{\mathbf{3}}^n = \mathbf{1}^q \oplus \mathbf{2} \oplus \mathbf{6} \\
\widetilde{\mathbf{3}}^m \otimes \widetilde{\mathbf{3}}^n = \mathbf{1}^q \oplus \mathbf{2} \oplus \widetilde{\mathbf{3}} \oplus \widetilde{\mathbf{3}}' \\
\widetilde{\mathbf{3}}^m \otimes \overline{\mathbf{3}}^n = \mathbf{3}^p \oplus \mathbf{6} \\
\overline{\mathbf{3}}^m \otimes \overline{\mathbf{3}}^n = \mathbf{3} \oplus \mathbf{3}' \oplus \widetilde{\mathbf{3}}^p \\
\mathbf{3}^m \otimes \mathbf{6} = \mathbf{3} \oplus \widetilde{\mathbf{3}} \oplus \mathbf{3}' \oplus \widetilde{\mathbf{3}}' \oplus \mathbf{6} \\
\widetilde{\mathbf{3}}^m \otimes \mathbf{6} = \mathbf{3} \oplus \overline{\mathbf{3}} \oplus \mathbf{3}' \oplus \overline{\mathbf{3}}' \oplus \mathbf{6} \\
\overline{\mathbf{3}}^m \otimes \mathbf{6} = \widetilde{\mathbf{3}} \oplus \overline{\mathbf{3}} \oplus \widetilde{\mathbf{3}}' \oplus \overline{\mathbf{3}}' \oplus \mathbf{6} \\
\mathbf{6} \otimes \mathbf{6} = \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{2} \oplus \mathbf{2} \oplus \mathbf{3} \oplus \mathbf{3}' \oplus \widetilde{\mathbf{3}} \oplus \widetilde{\mathbf{3}}' \oplus \overline{\mathbf{3}} \oplus \overline{\mathbf{3}}' \oplus \mathbf{6} \oplus \mathbf{6}
\end{gathered}$$

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$$\begin{array}{llll}
 & S & T & U \\
\mathbf{1}: & 1 & 1 & 1 \\
\mathbf{1}': & 1 & 1 & -1 \\
\mathbf{2}: & \mathcal{I}_{2\times 2} & \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\mathbf{3}: & s_3 & t_3 & u_3 \\
\overline{\mathbf{3}}: & s_3 & t_3^* & u_3 \\
\mathbf{3}': & s_3 & t_3 & -u_3 \\
\overline{\mathbf{3}}': & s_3 & t_3^* & -u_3 \\
\widetilde{\mathbf{3}}: & \mathcal{I}_{3\times 3} & t_3 & v s_3 \\
\widetilde{\mathbf{3}}': & \mathcal{I}_{3\times 3} & t_3 & -v s_3 \\
\mathbf{6}: & \begin{pmatrix} s_3 & 0 \\ 0 & s_3 \end{pmatrix} & \begin{pmatrix} t_3 & 0 \\ 0 & t_3 \end{pmatrix} & \begin{pmatrix} 0 & w \\ w^* & 0 \end{pmatrix}
\end{array}$$

$$s_3 = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad u_3 = \frac{1}{3}\begin{pmatrix} -1+\sqrt{3} & -1-\sqrt{3} & -1 \\ -1-\sqrt{3} & -1 & -1+\sqrt{3} \\ -1 & -1+\sqrt{3} & -1-\sqrt{3} \end{pmatrix},$$

$$s_3 u_3 = \frac{1}{3}\begin{pmatrix} -1-\sqrt{3} & \sqrt{3}-1 & -1 \\ \sqrt{3}-1 & -1 & -1-\sqrt{3} \\ -1 & -1-\sqrt{3} & \sqrt{3}-1 \end{pmatrix}, \quad \text{and} \quad t_3 = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix}.$$

de Adelhart Toorop, Hagedorn, Feruglio; Ding

Bi-Trimaximal neutrino mixing

Klein Symmetry in $\Delta(96)$:

$$S = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \; U = \frac{1}{3}\begin{pmatrix} -1+\sqrt{3} & -1-\sqrt{3} & -1 \\ -1-\sqrt{3} & -1 & -1+\sqrt{3} \\ -1 & -1+\sqrt{3} & -1-\sqrt{3} \end{pmatrix} \qquad T = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix}$$

$$SM^\nu S = M^\nu \quad TM^E T = M^E \quad UM^\nu U = M^\nu$$

St. George's Cross

$$U_{\rm BT} = \begin{pmatrix} a_+ & \frac{1}{\sqrt{3}} & a_- \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ a_- & -\frac{1}{\sqrt{3}} & a_+ \end{pmatrix} P, \qquad a_\pm = (1 \pm \tfrac{1}{\sqrt{3}})/2$$

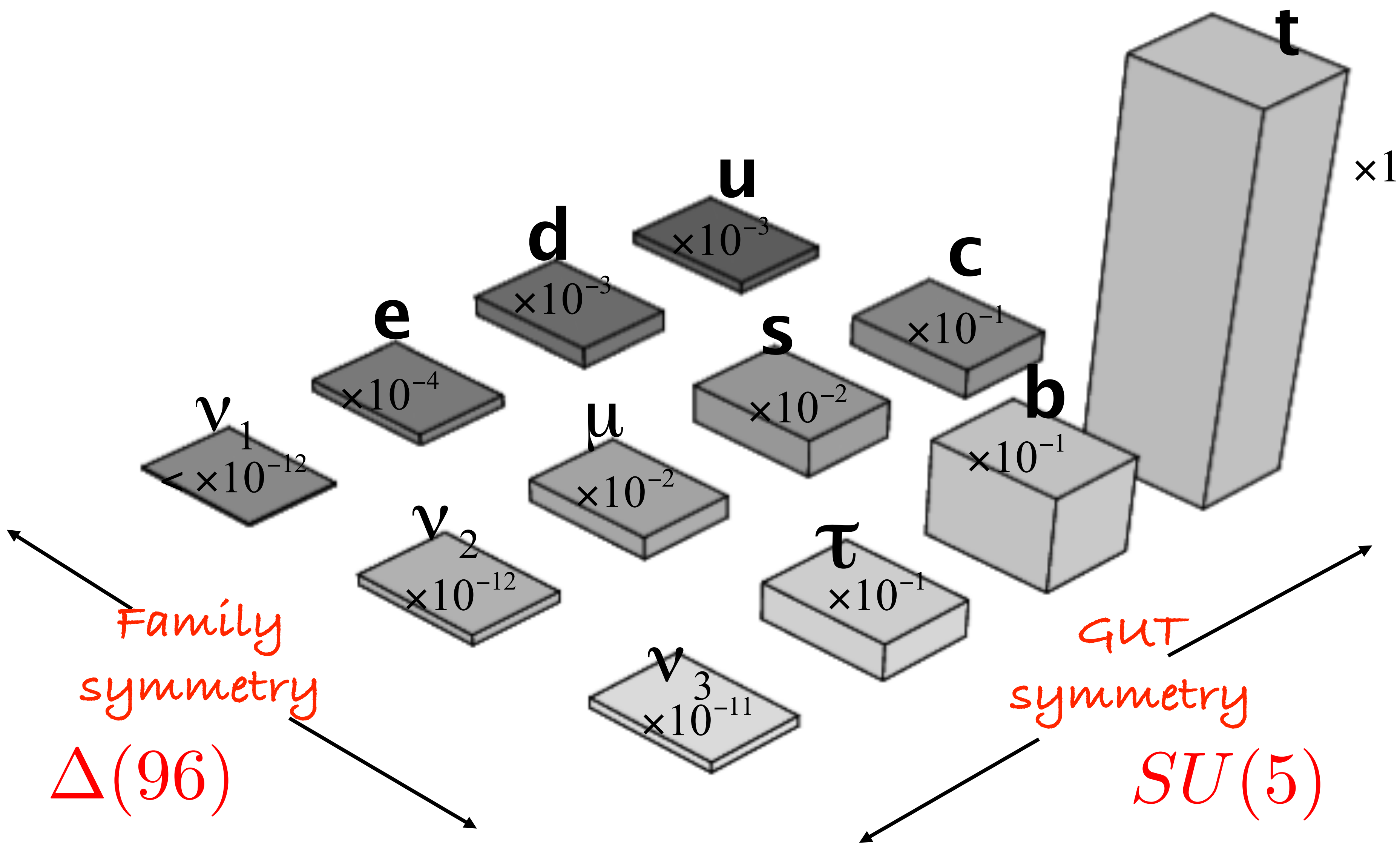
$$\sin\theta_{12} = \sin\theta_{23} = \sqrt{\tfrac{8-2\sqrt{3}}{13}} \approx 0.591 \qquad (\theta_{12} = \theta_{23} \approx 36.2^\circ)$$
$$\sin\theta_{13} = a_- \approx 0.211 \qquad (\theta_{13} \approx 12.2^\circ).$$

$$s \approx 0.023, \quad a \approx -0.165, \quad r \approx 0.299.$$

Data

$$s = -0.03 \pm 0.03, \qquad a = -0.10 \pm 0.03, \qquad r = 0.22 \pm 0.01$$

Family Symmetry × GUTs



$\Delta(96)\times SU(5)$

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Flavons

Field	T_3	T	F	N	$H_{5,\bar{5}}$	$H_{\overline{45}}$	Φ^u_2	$\bar{\Phi}^u_2$	$\Phi^d_{\bar{3}}$	$\bar{\Phi}^d_{\bar{3}}$	Φ^d_2	$\Phi^\nu_{\bar{3}'}$	$\Phi^\nu_{\tilde{3}'}$	$\Phi^\nu_{\tilde{3}}$
$SU(5)$	$\mathbf{10}$	$\mathbf{10}$	$\bar{\mathbf{5}}$	$\mathbf{1}$	$\mathbf{5},\bar{\mathbf{5}}$	$\overline{\mathbf{45}}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$\Delta(96)$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{3}$	$\bar{\mathbf{3}}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{2}$	$\bar{\mathbf{3}}$	$\bar{\mathbf{3}}$	$\mathbf{2}$	$\bar{\mathbf{3}}'$	$\widetilde{\mathbf{3}}'$	$\widetilde{\mathbf{3}}$
$U(1)$	0	x	y	$-y$	0	z	$-2x$	0	$-y$	$-x-y-2z$	z	$2y$	$2y$	w
Z_3	1	1	ω^2	ω	$1,\omega$	ω	1	1	1	1	1	ω	ω	ω

Yukawa Operators

$$y_u T_3 T_3 H_5 + y'_u \frac{1}{M} T T \Phi^u_2 H_5 + y''_u \frac{1}{M^2} T T \Phi^u_2 \bar{\Phi}^u_2 H_5,$$

Up

Down

$$y_d \frac{1}{M} F T_3 \Phi^d_{\bar{3}} H_{\bar{5}} + y'_d \frac{1}{M^2} (F \bar{\Phi}^d_{\bar{3}})_1 (T \Phi^d_2)_1 H_{\overline{45}} + y''_d \frac{1}{M^3} (F \Phi^d_2 \Phi^d_2)_3 (T \bar{\Phi}^d_{\bar{3}})_{\bar{3}} H_{\bar{5}},$$

Georgi-Jarlskog

and Charged Lepton

$$y_D F N H_5 + \overline{y}_M N N \Phi^\nu_{\bar{3}'} + \widetilde{y}_M N N \Phi^\nu_{\tilde{3}'},$$

Neutrino

Δ(96)×SU(5)

Driving Fields

Field	X_1^ν	X_2^ν	X_6^ν	X_1^d	Y_1^d	Z_1^d	X_1^u	X_1^{ud}	$X_{1'}^{\nu d}$	X_2^{du}
$\Delta(96)$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{6}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1'}$	$\mathbf{2}$
$U(1)$	$-6y$	$-4y$	$-2y-w$	$4y$	$-2z$	$x+3y+z$	$2x$	$2x+4y$	$x+2y+2z-w$	$2x-z$
Z_3	1	ω	ω	1	1	1	1	1	ω^2	1

Flavon Superpotential

$$X_6^\nu \Phi^\nu_{\tilde{3}} \Phi^\nu_{\bar{3}'} \qquad X_2^\nu \Phi^\nu_{\tilde{3}'} \Phi^\nu_{\tilde{3}'}$$

$$\frac{1}{M} X_1^\nu \left[g_0 \Phi^\nu_{\bar{3}'} \Phi^\nu_{\bar{3}'} \Phi^\nu_{\bar{3}'} + g_1 \Phi^\nu_{\bar{3}'} \Phi^\nu_{\bar{3}'} \Phi^\nu_{\tilde{3}'} + g_2 \Phi^\nu_{\tilde{3}'} \Phi^\nu_{\tilde{3}'} \Phi^\nu_{\tilde{3}'} \right]$$

$$Y_1^d \Phi_2^d \Phi_2^d \qquad X_2^{du} \Phi_2^d \Phi_2^u \qquad X_1^u \Phi_2^u \bar{\Phi}_2^u$$

$$\frac{1}{M^2} X_1^d \Phi^d_{\bar{3}} \Phi^d_{\bar{3}} \Phi^d_{\bar{3}} \Phi^d_{\bar{3}} + \frac{1}{M^3} X_1^{ud} \Phi_2^u \Phi^d_{\bar{3}} \Phi^d_{\bar{3}} \Phi^d_{\bar{3}} \Phi^d_{\bar{3}}$$

$$\frac{1}{M} X_{1'}^{\nu d} \Phi^\nu_{\tilde{3}} \Phi^d_{\bar{3}} \bar{\Phi}^d_{\bar{3}} + \frac{1}{M^2} Z_1^d \Phi_2^d \Phi^d_{\bar{3}} \Phi^d_{\bar{3}} \bar{\Phi}^d_{\bar{3}}$$

Flavon Alignments

$$\langle \Phi^\nu_{\bar{3}'} \rangle = \varphi^\nu_{\bar{3}'} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \text{and} \quad \langle \Phi^\nu_{\tilde{3}'} \rangle = \varphi^\nu_{\tilde{3}'} \begin{pmatrix} v_1 \\ \frac{1}{2}(v_1+v_3) \\ v_3 \end{pmatrix}$$

$$\langle \Phi_2^u \rangle = \varphi_2^u \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \langle \bar{\Phi}_2^u \rangle = \bar{\varphi}_2^u \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\langle \Phi_2^d \rangle = \varphi_2^d \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \langle \Phi^d_{\bar{3}} \rangle = \varphi^d_{\bar{3}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad \langle \bar{\Phi}^d_{\bar{3}} \rangle = \bar{\varphi}^d_{\bar{3}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Quark and Lepton Mass Matrices

$$\varphi_2^u/M \approx \lambda^4 \quad \bar{\varphi}_2^u/M \approx \lambda^4 \quad \varphi_2^d/M \approx \lambda\,, \quad \varphi_3^d/M \approx \lambda^{1+k}\,, \quad \bar{\varphi}_3^d/M \approx \lambda^{2+k}$$

$$k = 1$$

Up

$$M_u \approx v_u \begin{pmatrix} y_u''\bar{\varphi}_2^u\varphi_2^u/M^2 & 0 & 0 \\ 0 & y_u'\varphi_2^u/M & 0 \\ 0 & 0 & y_u \end{pmatrix} \quad M_u \sim v_u \begin{pmatrix} \lambda^8 & 0 & 0 \\ 0 & \lambda^4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Georgi-Jarlskog

$$m_e/m_d = 1/3, \quad m_\mu/m_s = 3, \quad m_\tau/m_b = 1\,,$$
$$m_d : m_s : m_b \approx \lambda^4 : \lambda^2 : 1,$$

Down

$$\theta_{12}^d \approx \sqrt{m_d/m_s}$$

$$M_d \approx v_d \begin{pmatrix} 0 & y_d''(\varphi_2^d)^2\bar{\varphi}_3^d/M^3 & -y_d''(\varphi_2^d)^2\bar{\varphi}_3^d/M^3 \\ y_d''(\varphi_2^d)^2\bar{\varphi}_3^d/M^3 & y_d'\varphi_2^d\bar{\varphi}_3^d/M^2 - y_d''(\varphi_2^d)^2\bar{\varphi}_3^d/M^3 & -y_d'\varphi_2^d\bar{\varphi}_3^d/M^2 \\ 0 & 0 & y_d\varphi_3^d/M \end{pmatrix} \quad M_d \sim v_d \begin{pmatrix} 0 & \lambda^5 & \lambda^5 \\ \lambda^5 & \lambda^4 & \lambda^4 \\ 0 & 0 & \lambda^2 \end{pmatrix}$$

zero 1-3 and 2-3

$$M_e \approx v_d \begin{pmatrix} 0 & y_d''(\varphi_2^d)^2\bar{\varphi}_3^d/M^3 & 0 \\ y_d''(\varphi_2^d)^2\bar{\varphi}_3^d/M^3 & -3y_d'\varphi_2^d\bar{\varphi}_3^d/M^2 - y_d''(\varphi_2^d)^2\bar{\varphi}_3^d/M^3 & 0 \\ -y_d''(\varphi_2^d)^2\bar{\varphi}_3^d/M^3 & 3y_d'\varphi_2^d\bar{\varphi}_3^d/M^2 & y_d\varphi_3^d/M \end{pmatrix} \quad M_e \sim v_d \begin{pmatrix} 0 & \lambda^5 & 0 \\ \lambda^5 & 3\lambda^4 & 0 \\ \lambda^5 & 3\lambda^4 & \lambda^2 \end{pmatrix}$$

Charged Lepton (violates T)

Neutrino (respects S,U)

$$M_D = y_D v_u \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad M_{Maj} = \overline{y}_M \varphi_{3'}^\nu \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} + \widetilde{y}_M \varphi_{3'}^\nu \begin{pmatrix} v_3 & v_1 & \frac{1}{2}(v_1+v_3) \\ v_1 & \frac{1}{2}(v_1+v_3) & v_3 \\ \frac{1}{2}(v_1+v_3) & v_3 & v_1 \end{pmatrix}$$

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Bi-Trimaximal neutrino mixing with charged lepton corrections ~~T~~

zero 1-3 and 2-3

St. George's Cross

$$U_{\rm BT} = \begin{pmatrix} a_+ & \frac{1}{\sqrt{3}} & a_- \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ a_- & -\frac{1}{\sqrt{3}} & a_+ \end{pmatrix} P, \qquad V_e \approx P' \begin{pmatrix} c^e_{12} & -s^e_{12}e^{-i\delta^e_{12}} & 0 \\ s^e_{12}e^{i\delta^e_{12}} & c^e_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$a_\pm = (1 \pm \tfrac{1}{\sqrt{3}})/2$$

$$U_{\rm PMNS} = V_{e_{\rm L}} V^\dagger_{\nu_{\rm L}}$$

$$U_{\rm PMNS} \approx P'' \begin{pmatrix} a_+c^e_{12} + \frac{1}{\sqrt{3}}s^e_{12}e^{-i\delta^e_{12}} & \frac{1}{\sqrt{3}}c^e_{12} - \frac{1}{\sqrt{3}}s^e_{12}e^{-i\delta^e_{12}} & a_-c^e_{12} - \frac{1}{\sqrt{3}}s^e_{12}e^{-i\delta^e_{12}} \\ a_+s^e_{12}e^{i\delta^e_{12}} - \frac{1}{\sqrt{3}}c^e_{12} & \frac{1}{\sqrt{3}}s^e_{12}e^{i\delta^e_{12}} + \frac{1}{\sqrt{3}}c^e_{12} & a_-s^e_{12}e^{i\delta^e_{12}} + \frac{1}{\sqrt{3}}c^e_{12} \\ a_- & -\frac{1}{\sqrt{3}} & a_+ \end{pmatrix} P$$

Predictions:

OK with data

$\theta^e_{12} \approx \lambda/3 \quad \delta^e_{12} \approx 0$

$$\tan\theta_{23} \approx \frac{\frac{1}{\sqrt{3}}c^e_{12} + a_-s^e_{12}}{a_+} \approx 0.750 \qquad \boxed{\theta_{23} \approx 36.9^\circ}$$

$$\sin\theta_{13} \approx a_- - \frac{1}{\sqrt{3}}\theta^e_{12}\cos\delta^e_{12} \qquad \boxed{\theta_{13} \approx 9.6^\circ}$$

$$\tan\theta_{12} \approx \frac{\frac{1}{\sqrt{3}}c^e_{12} - \frac{1}{\sqrt{3}}s^e_{12}}{a_+c^e_{12} + \frac{1}{\sqrt{3}}s^e_{12}} \approx 0.642 \qquad \boxed{\theta_{12} \approx 32.7^\circ}$$

Zero CP phase $\boxed{\delta \approx 0}$

Summary

- Simple patterns BM, TB, GR excluded, TBC mixing OK
- Expand about TB mixing, data prefers: $a<0$, $s<0$, $r=0.22$
- TM1: $s=0$, $a=r.\cos\delta$, TM2: $s=0$, $a=-r/2.\cos\delta$
- Two theory approaches: Symmetry or Anarchy
- Family Symmetry implemented indirectly or directly
- Indirect models: CSD→TB, PCSD→TBR, CSD2→TM1
- Direct models: A4,S4,A5→BM,TB,GR, u breaking→TM2
- Delta(96) → $\theta_{13}\sim 12^{\circ}$, $\theta_{12}=\theta_{23}\sim 36^{\circ}$ excluded
- Delta(96)xSU(5) → $\theta_{13}\sim 9.6^{\circ}$, $\theta_{12}\sim 32.7^{\circ}$, $\theta_{23}\sim 36.9^{\circ}$ OK

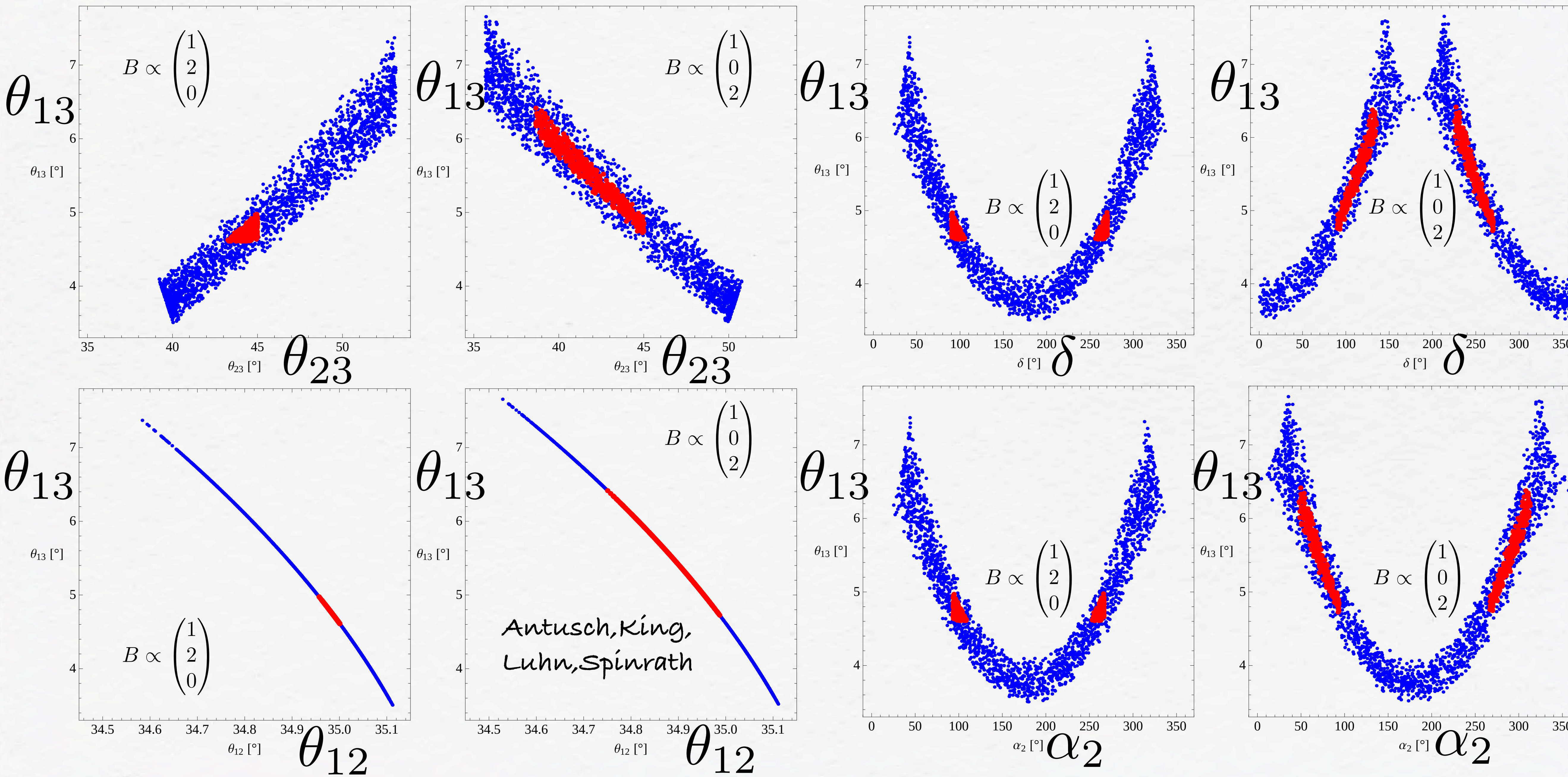
CSD2

$$r = \frac{2}{3}\frac{m_2^\nu}{m_3^\nu} \sim \frac{2}{15} \quad \rightarrow \quad \theta_{13} \sim 5^\circ - 6^\circ$$

Too small! Occam's razor fails

Charged lepton corrections? (in progress)

- Leptogenesis phase = Oscillation phase $Y_B \propto \pm \sin\delta$



Trimaximal2 from $\cancel{U}$

$$\frac{1}{3}\begin{pmatrix}-1 & 2 & 2\\ 2 & -1 & 2\\ 2 & 2 & -1\end{pmatrix} \quad \begin{pmatrix}1 & 0 & 0\\ 0 & \omega^2 & 0\\ 0 & 0 & \omega\end{pmatrix} \quad \mp\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$$

$$S, T, U \in S_4$$

- TB mixing respects Klein symmetry

$$SM^\nu S = M^\nu \quad TM^E T = M^E \quad UM^\nu U = M^\nu$$

- TM2 mixing respects discrete sym $S, T \in A_4$

$$SM^\nu S = M^\nu \quad TM^E T = M^E$$

U is broken

$$s \approx 0 \ , \qquad a \approx -\frac{1}{2} r \cos\delta$$

$$U_{TB} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} P, \qquad U_{\mathrm{TM}_2} = P' \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} r e^{-i\delta} \\ -\frac{1}{\sqrt{6}}(1+\frac{3}{2} r e^{i\delta}) & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}(1-\frac{1}{2} r e^{-i\delta}) \\ -\frac{1}{\sqrt{6}}(1-\frac{3}{2} r e^{i\delta}) & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}}(1+\frac{1}{2} r e^{-i\delta}) \end{pmatrix} P$$

This suggests that TM2 mixing can arise from A4 or S4 broken to A4 (breaking U)

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